Integral Solutions of Ablation Problems with Time-Dependent Heat Flux

T. F. Zien*
Naval Surface Weapons Center, White Oak Laboratory, Silver Spring, Md.

Approximate solutions are presented for the one-dimensional transient ablation problem with two specific forms of time-dependent boundary heat flux, i.e., $q_0 \sim t^m$ and $q_0 \sim e^t$, based on Landau's idealized ablation model. The solutions are obtained using the simple integral procedure employed eariler by the author for the solution of both boundary-layer flow problems and transient heat conduction problems. These solutions are compared with the corresponding solutions by the classical heat balance integral method. Some special features of the solutions are noted.

I. Introduction

TRANSIENT heat conduction in a solid undergoing phase change represents a problem area of great technological importance. Examples of its application include metal casting, ice formation, space vehicle ablation, etc., to name only a few. Such problems are inherently nonlinear and involve a moving boundary whose location is usually unknown in advance. Exact solutions to these problems are very difficult and necessarily require considerable numerical computations, even if a simplified model of the problem is used in the study.

Analytical approximation methods can be useful in providing approximate solutions to these problems without the expenditure of an undue amount of computational efforts. Yet, a true insight into the physical phenomena can be obtained in the solution process. One such method is the heat balance integral (HBI) method first used by Goodman.[1] A refinement of the HBI method was recently introduced by Zien[2-4] in his study of boundary-layer flows and transient heat conduction, and a significant improvement on the HBI method has been amply demonstrated. In particular, the idealized ablation problem proposed and numerically solved earlier by Landau[5] was solved approximately in Ref. 4 along with some other problems using Zien's procedure. The solution was shown to be very accurate, and the dependence of the solution on the assumed temperature profiles appears much weaker than that of the corresponding solution by the HBI method. These encouraging results provide the incentive to the present efforts in applying the method to more general heat conduction problems involving phase transitions.

In this paper, the present integral method is employed to provide approximate solutions to the generalized Landau ablation problem in which the applied heat flux is time-dependent. Solutions are presented for two specific cases of $q_0(t)$, i.e., $q_0 \sim t^m$ and $q_0 \sim e^t$. These cases are believed to simulate more realistically the ablation phenomena in the atmospheric entry environments than the constant boundary heat flux model. The case with a power-law heat flux has been studied earlier by Vallerani[6] using the HBI method, whereas the study of the exponential heat flux case does not seem to have been reported in the literature to the best of the author's knowledge.

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*Research Aerospace Engineer, Applied Mathematics Branch, Research and Technology Dept. Associate Fellow AIAA.

The solutions based on the present method are compared with the corresponding solutions based on the HBI method in this paper. Inasmuch as Vallerani's results[6] are presented only in graphical form, it was decided to recalculate his solutions in the present work for an accurate comparison. In addition, the HBI solutions of the exponential heat flux case are also carried out and included in the present paper. Tabulated solutions are available in Ref. 7.

It is hoped that the simple, approximate solutions presented here will be useful also as reference data for the elaborate numerical solutions of these difficult nonlinear problems of transient heat conduction, e.g., the work described in Ref. 8.

II. The Ablation Model

The model used here is a semi-infinite solid initially in a uniform temperature T_∞ lower than the phase-change temperature of the solid T_p. An unsteady heat flux $q_0(t)$ is then applied at the boundary until the boundary temperature reaches the phase-change temperature of the solid. This period is referred to as the preablation period. As the external heating continues, melting commences with the melting front progressing into the solid, and this period is referred to as the ablation period. In the idealized model it is assumed that the molten solid is removed instantaneously and completely upon its formation, say by the action of some aerodynamic forces, so that the melting line acts like a new (moving) boundary upon which the external heat flux $q_0(t)$ acts. This assumption is particularly appropriate for the ablation of subliming materials such as camphor, graphite, etc. Also, to simplify the calculations, the thermophysical properties of the solid are assumed constant. This model is the same as the one used earlier by Landau,[5] except that he considered only the special case of a constant q_0, and obtained solutions by entirely numerical means. The model is sketched in Fig. 1.

In terms of this idealized model, the governing equations and boundary conditions are as follows:

1) Preablation period

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad t_p > t > 0 \quad \infty > x > 0 \tag{1}$$

$$T(x,0) = T(\infty,t) = T_\infty \tag{2a}$$

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0} = q_0(t) \tag{2b}$$

where α and k are, respectively, the (constant) thermal diffusivity and heat conductivity of the solid. Also, t_p signifies the time at which the boundary temperature reaches T_p, i.e., $T(0,t_p) = T_p$.

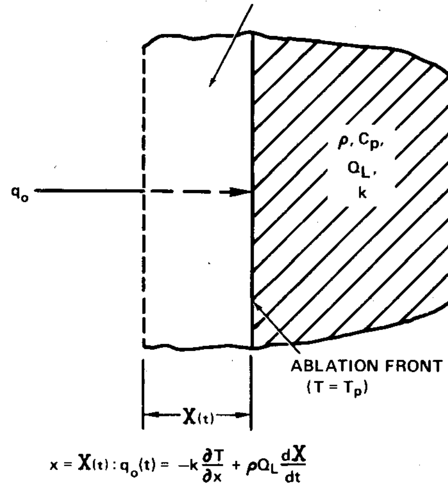


Fig. 1 Ablation model.

2) Ablation period

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2} \qquad \infty>t>t_p \quad \infty>x>X(t) \tag{3}$$

$$T(x,t_p^+)=T(x,t_p^-) \tag{4a}$$

$$T(X,t)=T_p \tag{4b}$$

$$T(\infty,t)=T_\infty \tag{4c}$$

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=X}+\rho Q_L\frac{\mathrm{d}X}{\mathrm{d}t}=q_0(t) \tag{4d}$$

Equation (4a) insures the continuity of the temperature distribution within the solid at the onset of ablation, $t=t_p$, and Eq. (4d) states the energy balance across the ablating front, $x=X(t)$. Note that the boundary condition, Eq. (4d), which relates the ablation speed, $\mathrm{d}X/\mathrm{d}t$, to the temperature gradient, $(\partial T/\partial x)_{x=X}$, is the basic source of nonlinearity of the problem.

III. Power-Law Boundary Heat Flux

A. Present Method

The θ-moment scheme[3,4] of the present integral method will be used in the calculations throughout this paper. Briefly speaking, the procedure makes combined use of the heat balance integral and the integral of the original heat equation after it is multiplied by $\theta(=T-T_\infty)$. A certain approximate temperature profile f is then substituted for the temperature in this integrated version of the heat equation. The heat balance integral based on the approximation temperature profile is used as the expression for the boundary heat flux.

Preablation Solution

Consider the idealized ablation problem with $q_0=At^m$, where A and m are constants. For the preablation period, an exponential profile for the temperature excess, $\theta=T-T_\infty$, is used in the calculation, i.e.,

$$f=\frac{q_0\delta}{k}\beta\exp\left(-\frac{x}{\delta}\right) \tag{5}$$

The profiles contains two parameters, δ and β, and satisfies only the boundary condition of $\theta_\infty=0$. Recall that the boundary flux is not to be obtained from $(\partial f/\partial x)_0$ in the present method.[3,4] The preablation solutions have already been presented in Ref. 3 and will be briefly summarized here.

The boundary temperature T_0 in dimensionless form is given as:

$$\Theta_0\equiv\frac{k(T_0-T_\infty)}{q_0\sqrt{\alpha t}}=\frac{\sqrt{m+5/4}}{(m+1)} \tag{6}$$

The parameters δ and β are:

$$\delta/\sqrt{\alpha t}=2/\sqrt{4m+5} \tag{7a}$$

$$\beta=(m+5/4)/(m+1) \tag{7b}$$

As is shown in Ref. 3, the boundary temperature as given by Eq. (6) agrees better than 1% with the exact solution for all m in the range $0\le m<\infty$.

Ablation starts when $T_0=T_p$. From Eq. (6), the corresponding time t_p is determined as

$$t_p=\left[\frac{k(m+1)(T_p-T_\infty)}{A\sqrt{(m+5/4)\alpha}}\right]^{\frac{1}{m+½}} \tag{8a}$$

The penetration depth δ at t_p follows from Eq. (7),

$$\delta_p=\left[\frac{k(m+1)(T_p-T_\infty)\alpha^m}{A\sqrt{m+5/4}}\right]^{\frac{1}{2m+1}}\Big/\sqrt{m+5/4} \tag{8b}$$

The quantities t_p and δ_p are conveniently used as the scales for time and length, respectively, in the formulation of the ablation problem. However, since t_p and δ_p just given are only approximate solutions dependent on the method of solution, it appears desirable to use the characteristic time and length of the problem, t_c and ℓ_c, as the scales for easy determination of the absolute accuracy of the solutions. From a simple dimensional consideration, it is easily found that t_c and ℓ_c of the problem are given as

$$t_c=[k(\Delta T)/A\sqrt{\alpha}]^{1/(m+½)} \tag{9a}$$

and

$$\ell_c=\sqrt{\alpha t_c}=[k(\Delta T)\alpha^m/A]^{1/(2m+1)} \tag{9b}$$

where $\Delta T\equiv T_p-T_\infty$.

Thus, we introduce the following two sets of dimensionless variables in the ensuing calculations:

$$\xi\equiv x/\delta_p \quad \tau\equiv t/t_p \quad \Delta\equiv\delta/\delta_p \quad \lambda\equiv x/\delta_p \tag{10a}$$

and

$$\xi_1\equiv x/\ell_c \quad \tau_1\equiv t/t_c \quad \Delta_1\equiv\delta/\ell_c \quad \lambda_1\equiv X/\ell_c \tag{10b}$$

Note, in particular, that at the onset of ablation, $\tau_p=1$ and

$$(\tau_{1_p})_Z=[(m+1)^2/(m+5/4)]^{1/(2m+1)} \tag{11}$$

The accuracy of the present preablation solution may also be determined on the basis of a comparison of τ_{1_p} as given by Eq. (11) with that given by the exact solution. The exact τ_{1_p} can be found in Ref. 9 as

$$(\tau_{1_p})_E=[\Gamma(m+3/2)/\Gamma(m+1)]^{1/(m+½)} \tag{12}$$

where Γ is the gamma function. It easily can be shown that τ_{1_p} of the present solution approaches the exact limit of $\tau_{1_p}=1$ as $m\to\infty$.

The comparison between the approximate solution and the exact solution for τ_{1_p} is shown in Fig. 2. The present solution is practically indistinguishable from the exact solution in the entire range of m, $\infty>m>0$ in the figure, the maximum error being about 1.8% at $m=0$. It is also interesting to note the existence of a maximum τ_{1_p} near $m=1.5$.

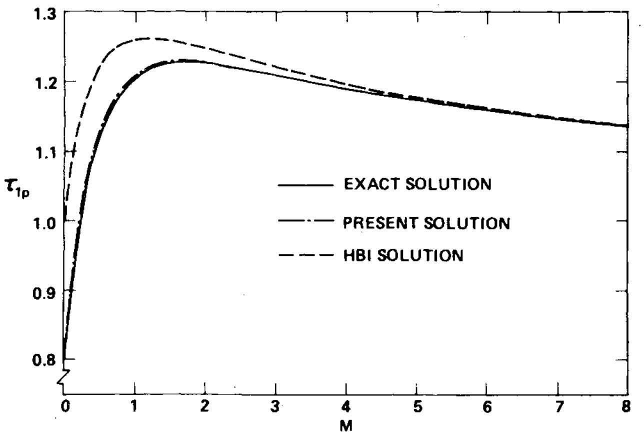


Fig. 2 Preablation time for $q_0 = At^m$.

Ablation Solution

The ablation problem is then formulated in dimensionless form by using dimensionless variables ξ, τ, Δ, and λ. The normalized temperature, $\theta = (T-T_\infty)/(T_p - T_\infty)$, is also used.

An exponential profile for θ is assumed

$$f = \exp\left[-\frac{x - X(t)}{\delta(t)}\right] \tag{13}$$

where $X(t)$ is the (unknown) ablation line location and $X(t_p) = 0$. Note that this choice of the temperature insures the continuity of the temperature field at $t = t_p$ if $\delta(t)$ is assumed to be continuous at t_p. However, $\delta(t)$, as noted here, no longer has the physical significance of a penetration depth, because at $x = \delta(t)$, f does not reduce to a value pertinent to thermal penetration.

The differential equation and boundary conditions, Eqs. (3) and (4), reduce to

$$\frac{\partial\theta}{\partial\tau} = (m+5/4)\frac{\partial^2\theta}{\partial\xi^2} \qquad \infty > \tau > 1 \quad \infty > \xi > \lambda(\tau) \tag{14}$$

$$\theta(\lambda,\tau) = 1 \tag{15a}$$

$$\theta(\infty,\tau) = 0 \tag{15b}$$

$$-(m+5/4)\frac{\partial\theta}{\partial\xi}\bigg|_\lambda = (m+1)\tau^m - \nu\frac{d\lambda}{d\tau} \tag{15c}$$

where $\nu = Q_L/c_p(T_p - T_\infty)$. Q_L is the latent heat of ablation per unit mass and c_p is the specific heat of the solid.

The integration of Eq. (14) from $\xi = \lambda$ to $\xi = \infty$, using f in Eq. (13) for θ, gives

$$\frac{d\Delta}{d\tau} + \frac{d\lambda}{d\tau} = -(m+5/4)\frac{\partial\theta}{\partial\xi}\bigg|_\lambda \tag{16}$$

The integration of Eq. (14) after it is multiplied by θ, again using f for θ, gives

$$\frac{1}{2}\frac{d\Delta}{d\tau} + \frac{d\lambda}{d\tau} = -(m+5/4)\left[2\frac{\partial\theta}{\partial\xi}\bigg|_\lambda + \frac{1}{\Delta}\right] \tag{17}$$

In our procedure, Eqs. (15c), (16), and (17) form the system for the three unknowns, λ, Δ, and the heat flux at the ablating front, $-\partial\theta/\partial\xi|_\lambda$.

Equations (15c) and (16) can be combined to give

$$\frac{d\Delta}{d\tau} + (1+\nu)\frac{d\lambda}{d\tau} = (m+1)\tau^m \tag{18}$$

which can be integrated to yield the integral

$$\lambda = (\tau^{m+1} - \Delta)/(1+\nu) \tag{19}$$

Here the initial conditions $\lambda(1) = 0$ and $\Delta(1) = 1$ have been used.

The system, Eqs. (15c, 16, and 17) can be reduced to the following single ordinary differential equation for Δ, viz.,

$$(1+3\nu)\frac{d\Delta}{d\tau} = -2(m+1)\tau^m + 2(m+5/4)(1+\nu)\frac{1}{\Delta} \tag{20}$$

with $\Delta(1) = 1$. The ablation thickness, λ, is then obtained readily from Eq. (19).

Equation (20) can be integrated in closed form only for the case $m = 0$, i.e., the case of a constant heat flux.[4] For other values of m, Eq. (20) can be integrated numerically with ease.

The behavior of the solution $\Delta(\tau)$ as $\tau \to \infty$ can be studied analytically. As $\tau \to \infty$, Eq. (20) suggests that the only self-consistent behavior of Δ is

$$\tau \to \infty: \Delta \to \Delta_\infty = (1+\nu)(m+5/4)\tau^{-m}/(m+1) \tag{21a}$$

and, consequently, from Eq. (19),

$$\tau \to \infty: \lambda \to \lambda_\infty = \tau^{m+1}/(1+\nu) \tag{21b}$$

It then follows that the asymptotic behavior of the ablation speed, $V^* = d\lambda/d\tau$, is

$$\tau \to \infty: V^* \to V^*_\infty = \tau^m(m+1)/(1+\nu) \tag{21c}$$

It is convenient to express the solutions, λ and V^*, in their normalized form, i.e.,

$$\lambda_n \equiv \lambda/\lambda_\infty$$

$$V^*_n = V^*/V^*_\infty$$

Converting the solutions to the dimensionless variables based on the basic scales [Eq. (10b)], we use the following simple conversion formulas:

$$\tau_1 = [(m+1)/\sqrt{m+5/4}]^{1/(m+1/2)}\tau \tag{22a}$$

$$\lambda_1 = [(m+1)/(m+5/4)^{m+1}]^{1/(2m+1)}\lambda \tag{22b}$$

$$V^*_1 = [(m+1)(m+5/4)^m]^{-1/(2m+1)}V^* \tag{22c}$$

where $V^*_1 \equiv d\lambda_1/d\tau_1$. Also, we have

$$\lambda_{1\infty} = \tau_1^{m+1}/(m+1)(1+\nu) \tag{22d}$$

and

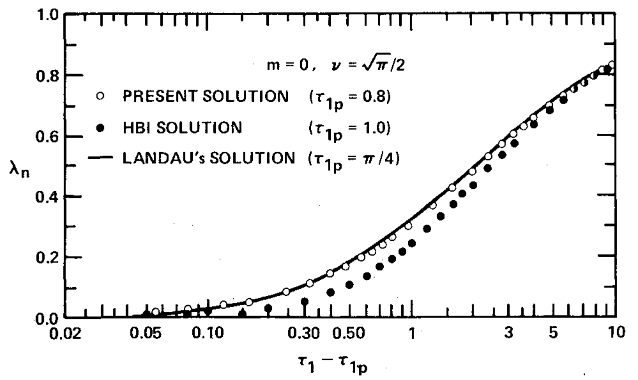
$$V^*_{1\infty} = \tau_1^m/(1+\nu) \tag{22e}$$

Obviously, the normalized λ and V^* remain invariant in the conversion, i.e., $\lambda_{1n} = \lambda_n$ and $V^*_{1n} = V^*_n$.

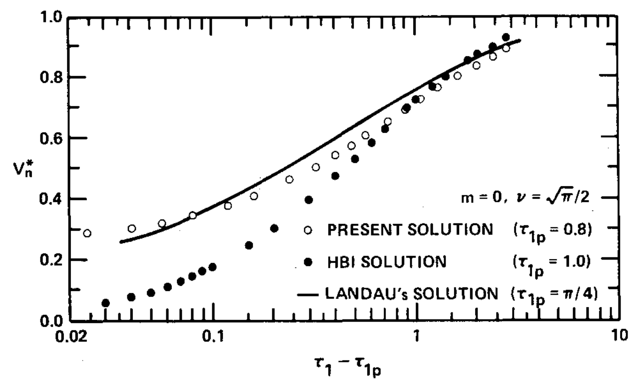
Results of λ_n and V^*_n are presented in Figs. 3-7 for some representative values of $(m,\nu) = (0,\sqrt{\pi}/2)$, (1,0.1), (1,1), (3,0.1), (3,1).

The case $(m,\nu) = (0,\sqrt{\pi}/2)$ is included here mainly because Landau's numerical solution[5] is available for comparison. This case has already been treated in Ref. 4 using different variables and is repeated here for easy reference. We note that the time scale used in these figures is measured from the onset of ablation, i.e., $(\tau_1 - \tau_{1p})$.

In comparison with Landau's solutions,[5] which appear only in graphical form, we first transform Landau's solutions to the present variables τ_1, λ_n, and V^*_n, and then carefully replot the solutions using an interpolator. Note that the



a) Ablation thickness



b) Ablation speed

Fig. 3 Ablation for $q_0 = A$, $\nu = \sqrt{\pi}/2$.

dimensionless time in Landau's solutions is equivalent to $(4/\pi)(\tau_l - \tau_{lp})$, the dimensionless speed is the same as V_n^*, the dimensionless ablation thickness is $(4/\pi)(1+\nu)\lambda_l$ and $\tau_{lp} = \pi/4$ (exact solution, Eq. (12)).

B. Classical HBI Method

Solutions based on the HBI method are presented by Vallerani.[6] Only the results are summarized here using our present notations.

The preablation solution expressed in terms of the preablation time, τ_{lp}, is

$$(\tau_{lp})_{\mathrm{HBI}} = (m+1)^{1/(2m+1)} \tag{23}$$

and is shown in comparison with $(\tau_{lp})_Z$ and $(\tau_{lp})_E$ in Fig. 2. Significant error in $(\tau_{lp})_{\mathrm{HBI}}$ is evident in the range $0 \le m \le 2$. Since the ablation solution depends on the preablation solution, this demonstrated error must be properly accounted for in assessing the accuracy of the ablation solutions where exact solutions are not available for comparison.

In the ablation period, the basic differential equation is

$$\frac{\mathrm{d}\Delta}{\mathrm{d}\tau} = \frac{m+1}{\nu}\left[(1+\nu)\frac{1}{\Delta} - \tau^m\right] \tag{24}$$

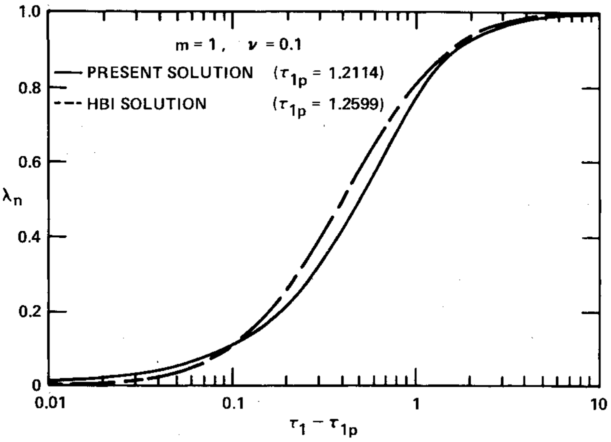
with $\Delta(1) = 1$.

The ablation thickness, λ, is related to Δ

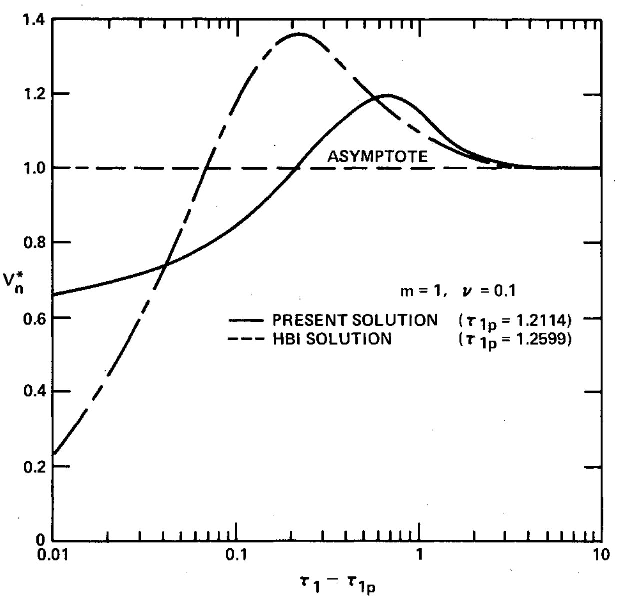
$$\lambda = (\tau^{m+1} - \Delta)/(1+\nu) \tag{25}$$

The asymptotic values of the dimensionless variables based on the basic scales are determined as (see Ref. 7 for details)

$$\lambda_{l\infty} = \tau_l^{m+1}/(m+1)(1+\nu) \tag{26a}$$



a) Ablation thickness



b) Ablation speed

Fig. 4 Ablation for $q_0 = At$, $\nu = 0.1$.

and

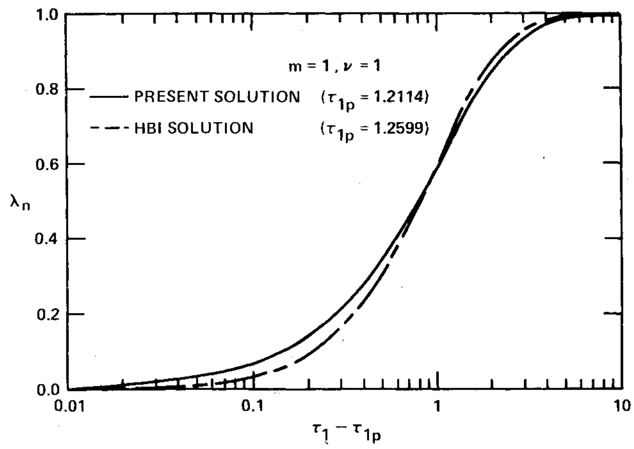
$$V_{l\infty}^* \equiv \left(\frac{\mathrm{d}\lambda_l}{\mathrm{d}\tau_l}\right)_{\tau_l=\infty} = \frac{\tau_l^m}{(1+\nu)} \tag{26b}$$

which agree with those of the new integral solutions, Eqs. (22d) and (22e). These asymptotic values are used to form the normalized quantities, λ_n and V_n^*, as before.

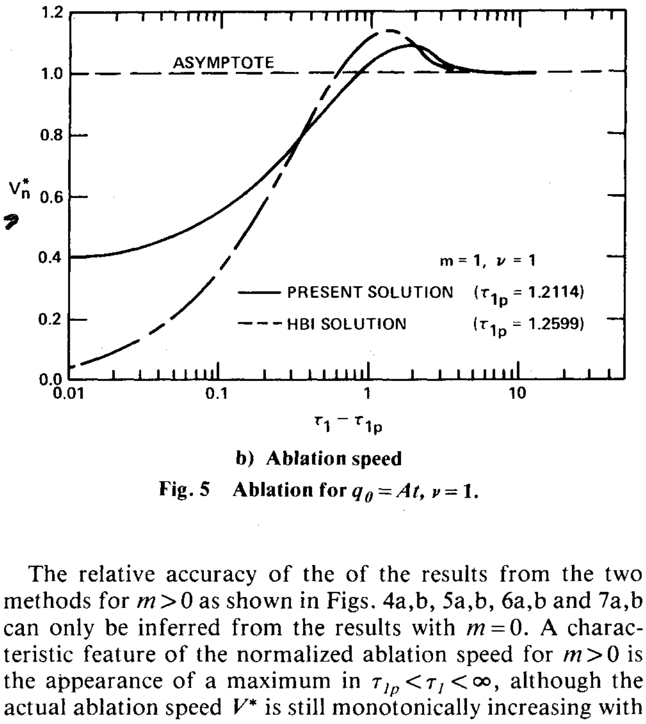
The results are shown in comparison with the new integral solutions of Sec. III A in Figs. 3-7.

C. Discussion

The solutions for $(m\nu) = (0, \sqrt{\pi}/2)$ are shown in Figs. 3a and 3b, where Landau's numerical solutions are also included to establish the accuracy of the approximate solutions of the two integral methods. It is clear that the present solutions are considerably more accurate than the HBI solutions both in λ_n and in V_n^*, although both integral solutions approach the exact limit as $\tau_l \to \infty$. As was pointed out earlier by Zien,[4] although the present solutions fail to give a zero initial ablation speed, their accuracy improves rapidly as ablation commences. The accuracy of the ablation speed predicted by the HBI method is clearly unsatisfactory in the time period $10^{-2} < \tau_l - \tau_{lp} < 1$. It should be noted that the error would be amplified if the plots were made with the absolute time τ_l, in view of the additional error of $(\tau_{lp})_{\mathrm{HBI}}$ discussed earlier.

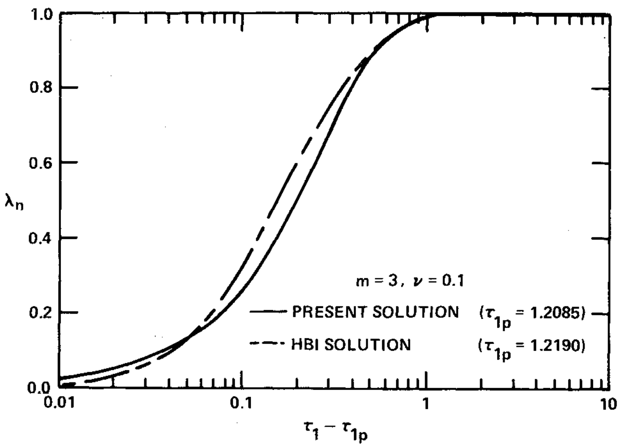


a) Ablation thickness

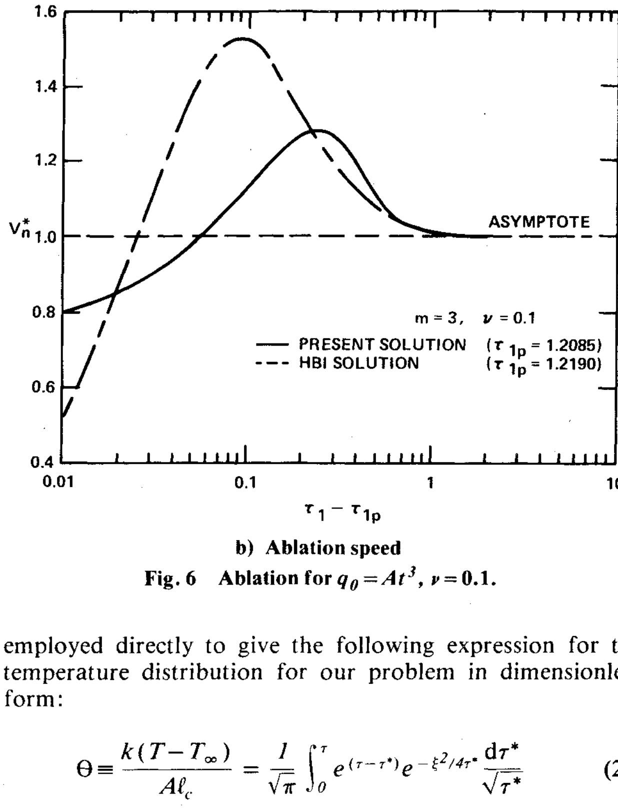


b) Ablation speed

Fig. 5 Ablation for $q_0 = At$, $\nu = 1$.



a) Ablation thickness



b) Ablation speed

Fig. 6 Ablation for $q_0 = At^3$, $\nu = 0.1$.

The relative accuracy of the of the results from the two methods for $m>0$ as shown in Figs. 4a,b, 5a,b, 6a,b and 7a,b can only be inferred from the results with $m=0$. A characteristic feature of the normalized ablation speed for $m>0$ is the appearance of a maximum in $\tau_{1p} < \tau_1 < \infty$, although the actual ablation speed V^* is still monotonically increasing with time for $m \geq 0$. It is also interesting to point out that for $m>0$, the predicted Δ_1 first increases and then decreases with increasing τ_1 and approaches zero like τ_1^{-m} in the limit. This result suggests that the local temperature far away from the ablation front $(\xi - \lambda \gg 1)$ may decrease with time at sufficiently large times before rising again. However, the dimensionless temperature and its time rate of change are too small $\{\sim \exp[-(\xi-\lambda)/\Delta]\}$ to be of any significance.

These interesting features are common to both types of integral solutions, and their confirmation must await the results of exact calculations.

IV. Exponential Boundary Heat Flux

A. Present Method

Again, the θ-moment scheme is used in the present solution, and the problem is still divided into a preablation period and an ablation period.

Preablation Solution

Consider now the case where $q_0 = Ae^{t/t_c}$, where A and t_c are constants. The characteristic time scale t_c and the characteristic length $\ell_c = \sqrt{\alpha t_c}$ will be used to form dimensionless variables in the ensuing analysis.

First, we derive the exact solution to the preablation problem. It is well known that Duhamel's integral can be employed directly to give the following expression for the temperature distribution for our problem in dimensionless form:

$$\Theta \equiv \frac{k(T-T_\infty)}{A\ell_c} = \frac{1}{\sqrt{\pi}} \int_0^\tau e^{(\tau-\tau^*)} e^{-\xi^2/4\tau^*} \frac{d\tau^*}{\sqrt{\tau^*}} \tag{27}$$

where $\tau \equiv t/t_c$ and $\xi \equiv x/\ell_c$. The boundary temperature Θ_0 is thus

$$\Theta_0 = \frac{1}{\sqrt{\pi}} e^\tau \int_0^\tau e^{-\tau^*} \frac{d\tau^*}{\sqrt{\tau^*}} = e^\tau \operatorname{erf}(\sqrt{\tau}) \tag{28}$$

where erf is the error function. In particular, when $T_0 = T_p$,

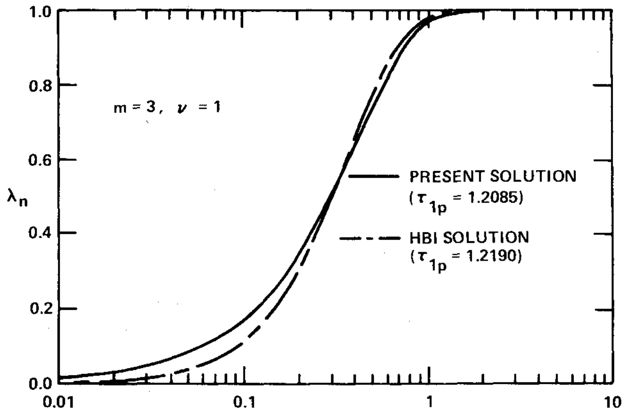
$$\Theta_{0p} = e^{\tau_p} \operatorname{erf}(\sqrt{\tau_p}) \tag{29}$$

Note that $\Theta_{0p} \to (2/\sqrt{\pi})\sqrt{\tau_p}$ as $\tau_p \to 0$ and $\Theta_{0p} \to e^{\tau_p}$ as $\tau_p \to \infty$.

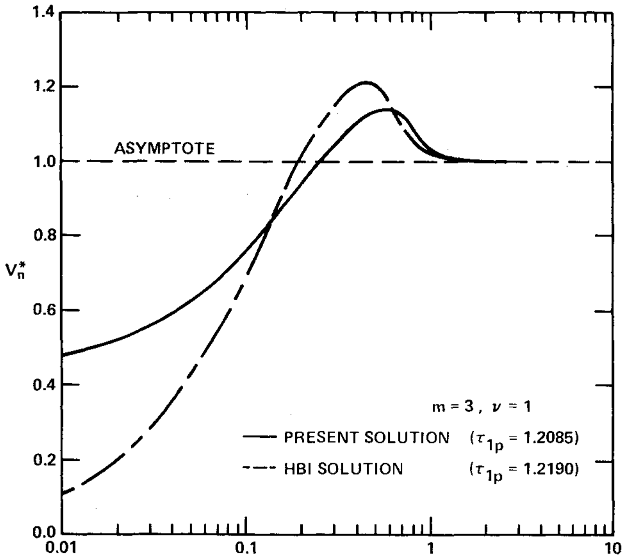
We then proceed to the approximate solution using the present method. Let the temperature excess, $\theta = T - T_\infty$, be expressed by the same exponential profile as in the previous case

$$f = \frac{q_0 \delta}{k} \beta \exp\left(-\frac{x}{\delta}\right) \tag{30}$$

It will be shown that for this case, β is no longer a constant parameter. The integration of the heat equation and the θ-



a) Ablation thickness



b) Ablation speed

Fig. 7 Ablation for $q_0 = At^3$, $\nu = 1$.

moment equation, respectively, yields the following equations for the solution of $\Delta(\tau)$ and $\beta(\tau)$ after some rearrangements:

$$\beta\Delta^2 = 1 - e^{-\tau} \tag{31}$$

and

$$(1-e^{\tau})\frac{d\beta}{d\tau} = 4\beta(1-\beta) + \beta e^{-\tau} \tag{32}$$

with $\Delta(0) = 0$ and $\beta(0) =$ bounded. Dimensionless variables are used here, $\tau \equiv t/t_c$ and $\Delta \equiv \delta/\sqrt{\alpha t_c}$.

Note that Eq. (32) can be recast into a linear equation for $1/\beta$ and easily solved in closed form. The complete solution is:

$$\Delta = \left(e^{4\tau} - \frac{16}{3}e^{3\tau} + 12e^{2\tau} - 16e^{\tau} + 4\tau + \frac{25}{3}\right)^{1/2} / (e^{\tau} - 1)^2 \tag{33}$$

$$\beta = e^{4\tau}(1-e^{-\tau})^5 \bigg/ \left(e^{4\tau} - \frac{16}{3}e^{3\tau} + 12e^{2\tau} - 16e^{\tau} + 4\tau + \frac{25}{3}\right) \tag{34}$$

The boundary temperature T_0 as a function of time then follows directly from Eq. (30). In particular, the dimensionless ablation temperature is given by

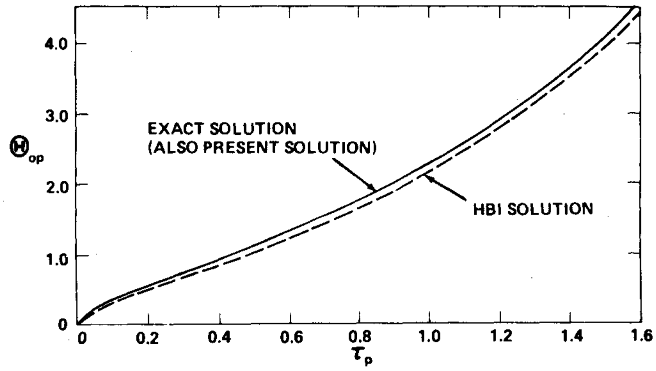


Fig. 8 Preablation boundary temperature for $q_0 = Ae^{\tau}$.

$$\Theta_{0p} = \frac{e^{3\tau_p}(1-e^{-\tau_p})^3}{\left(e^{4\tau_p} - \frac{16}{3}e^{3\tau_p} + 12e^{2\tau_p} - 16e^{\tau_p} + 4\tau_p + \frac{25}{3}\right)^{1/2}} \tag{35}$$

where τ_p is the preablation time. It can easily be shown that $\Theta_{0p} \to (\sqrt{5}/2)\sqrt{\tau_p}$ as $\tau_p \to 0$ and $\Theta_{0p} \to e^{\tau_p}$ as $\tau_p \to \infty$.

This result is plotted in Fig. 8. We note that the present solution is accurate to within 1%, and is virtually the same as the exact solution in the scale used in the plot.

Ablation Solution

The ablation problem with $q_0 = Ae^{\tau}$ is then formulated using the basic equation and boundary conditions given in Sec. III. Again, we introduce the dimensionless temperature θ,

$$\theta = \frac{T - T_\infty}{T_p - T_\infty} \tag{36}$$

and assume an exponential profile for θ as

$$f = \exp\left[-\frac{x - X(t)}{\delta(t)}\right] \tag{37}$$

where, as before $X(t)$ denotes the (unknown) ablation thickness ($X(t_p) = 0$) and the continuity of temperature field at $t = t_p$ is insured. The integration of the original heat equation and the θ-moment of the heat equation from $x = X$ to $x = \infty$ gives, respectively,

$$\frac{d\Delta}{d\tau} + \frac{d\lambda}{d\tau} = -\frac{\partial\theta}{\partial\xi}\bigg|_{\lambda} \tag{38}$$

and

$$\frac{1}{2}\frac{d\Delta}{d\tau} + \frac{d\lambda}{d\tau} = -2\frac{\partial\theta}{\partial\xi}\bigg|_{\lambda} - \frac{1}{\Delta} \tag{39}$$

where f is used for θ, and ξ, Δ, and λ, are, respectively, the dimensionless forms of x, δ, and X based on the length scale $\sqrt{\alpha t_c}$.

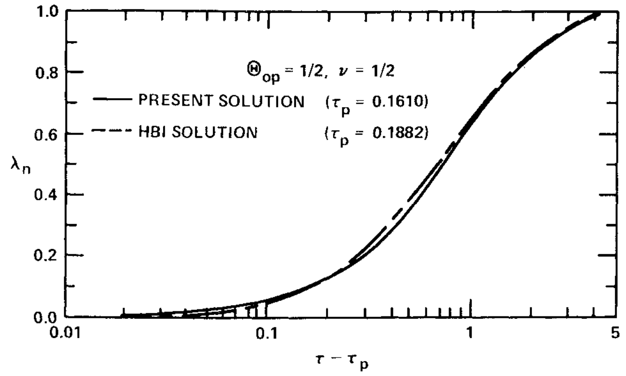
The ablation line boundary condition has the form

$$\frac{1}{\Theta_{0p}}e^{\tau} = -\frac{\partial\theta}{\partial\xi}\bigg|_{\lambda} + \nu\frac{d\lambda}{d\tau} \tag{40}$$

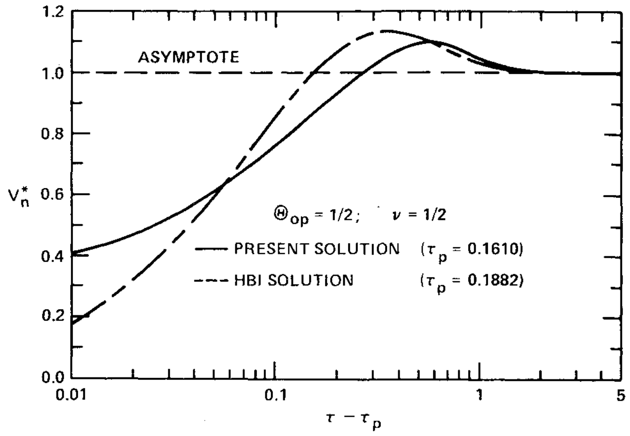
with $\nu \equiv Q_L/c_p(T_p - T_\infty)$.

Equations (38-40) form a system for the solution of Δ, λ, and $-\partial\theta/\partial\xi|_{\lambda}$. The system can be reduced to a single differential equation for Δ, i.e.,

$$\frac{d\Delta}{d\tau} = \frac{2}{1+3\nu}\left(-\frac{1}{\Theta_{0p}}e^{\tau} + \frac{1+\nu}{\Delta}\right) \tag{41}$$

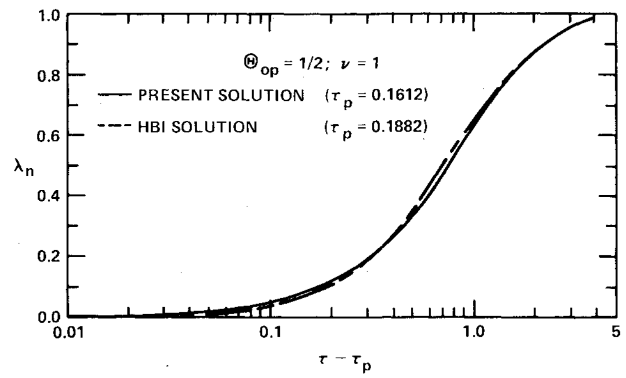


a) Ablation thickness

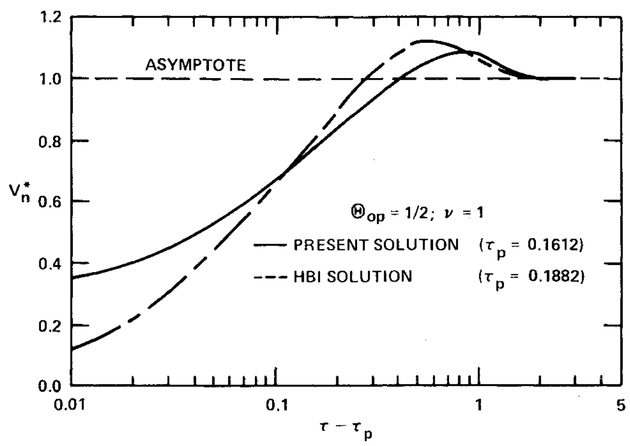


b) Ablation speed

Fig. 9 Ablation for $q_0 = Ae^{\tau}$, $\Theta_{0p} = 1/2$, $\nu = 1/2$.



a) Ablation thickness



b) Ablation speed

Fig. 10 Ablation for $q_0 = Ae^{\tau}$, $\Theta_{0p} = 1/2$, $\nu = 1$.

with $\Delta(\tau_p) = \Delta_p$ where Δ_p is obtained from the preablation solution, Eq. (33), for $\tau = \tau_p$. In a given problem, Θ_{0p} is specified and τ_p is to be obtained from Eq. (35). Numerical integration of Eq. (41) has been carried out for a range of values of (Θ_{0p}, ν).

The ablation thickness λ follows from the algebraic equation,

$$\lambda = \frac{1}{1+\nu}\left[\frac{1}{\Theta_{0p}}(e^{\tau} - e^{\tau_p}) - (\Delta - \Delta_p)\right] \tag{42}$$

The behavior of Δ as $\tau \to \infty$ can be deduced from Eq. (41) as

$$\Delta \to \Delta_\infty = \Theta_{0p}(1+\nu)e^{-\tau} \quad \text{as } \tau \to \infty \tag{43a}$$

Therefore, λ behaves as

$$\lambda \to \lambda_\infty = e^{\tau}/(1+\nu)\Theta_{0p} \quad \text{as } \tau \to \infty \tag{43b}$$

The normalized value of λ is introduced, i.e.,

$$\lambda_n = \lambda/\lambda_\infty = 1 - \exp[-(\tau - \tau_p)] - \Theta_{0p}(\Delta - \Delta_p)e^{-\tau} \tag{44}$$

The ablation speed, $V^* = d\lambda/d\tau$, can be found easily, and it is noted that the ablation speed behaves like $e^{\tau}/(1+\nu)\Theta_{0p}$ as $\tau \to \infty$, i.e., $V_\infty^* = \lambda_\infty$.

Some representative results for λ_n and V_n^* are shown in Figs. 9-12 for $(\Theta_{0p}, \nu) = (½,½)$, $(½,1)$, $(1,½)$, and $(1,1)$, respectively.

B. Classical HBI Method

The same problem can be studied by the standard HBI method. However, the results do not seem to have been reported in the literature. In the present work, the HBI solutions are also generated, using similar exponential temperature profiles, i.e.,

$$0 < t \le t_p: T - T_\infty = f = \frac{q_0\delta}{k}\exp\left(-\frac{x}{\delta}\right) \tag{45a}$$

$$t_p \le t < \infty: \frac{T - T_\infty}{T_p - T_\infty} = \exp\left(-\frac{x - X}{\delta}\right) \tag{45b}$$

The HBI procedure is well known, only the results will be presented here. For the preablation period, we have

$$\Delta = (1 - e^{-\tau})^{½} \tag{46}$$

and

$$\Theta_0 = e^{\tau}(1 - e^{-\tau})^{½} \tag{47}$$

Therefore,

$$\Theta_{0p} = e^{\tau_p}(1 - e^{-\tau})^{½} \tag{48}$$

so that $\Theta_{0p} \to \sqrt{\tau_p}$ as $\tau_p \to 0$ and $\Theta_{0p} \to e^{\tau_p}$ as $\tau_p \to \infty$.

Equation (48) is also included in Fig. 8 for comparison with the solutions by the present method. It is seen that Θ_{0p} from the classical HBI method is lower than that predicted by the present method. We recall that this trend is the same as that found in the case of a power-law heat flux.[4] The HBI solution is found to be in error by about 5-12% in the range $0.05 < \tau_p < 1.6$.

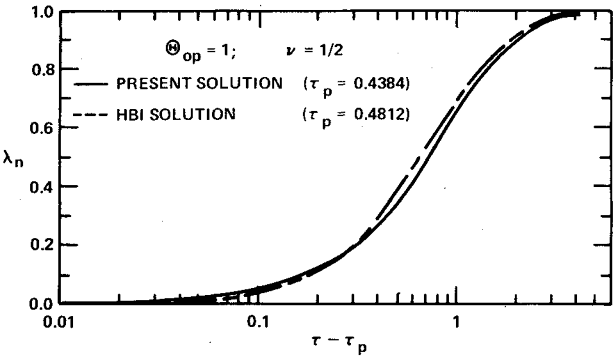
In the ablation phase, the system contains only two equations, viz.,

$$\frac{d\Delta}{d\tau} + \frac{d\lambda}{d\tau} = \frac{1}{\Delta} \tag{49}$$

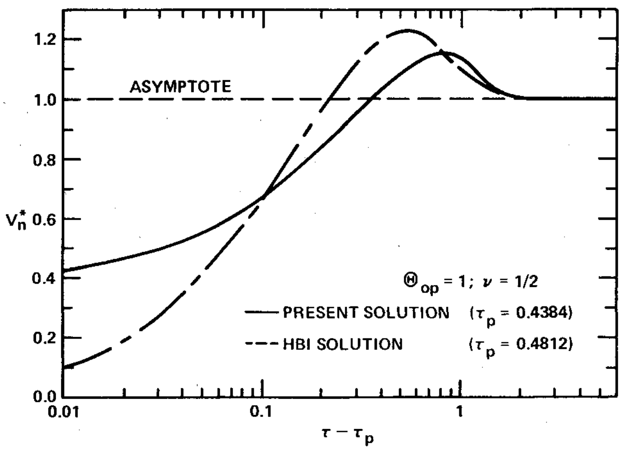
and

$$\frac{1}{\Theta_{0p}}e^{\tau} = \frac{1}{\Delta} + \nu\frac{d\lambda}{d\tau} \tag{50}$$

with $\Delta(\tau_p) = \Delta_p$ and $\lambda(\tau_p) = 0$.

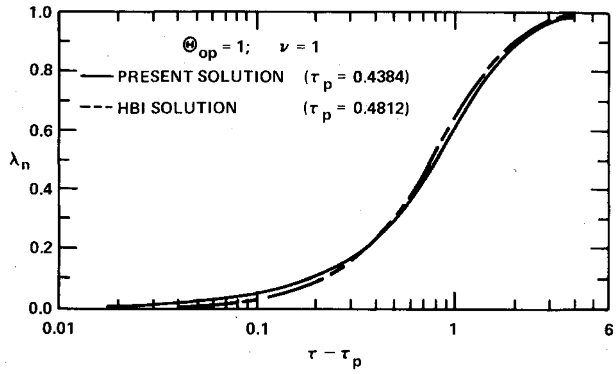


a) **Ablation thickness**

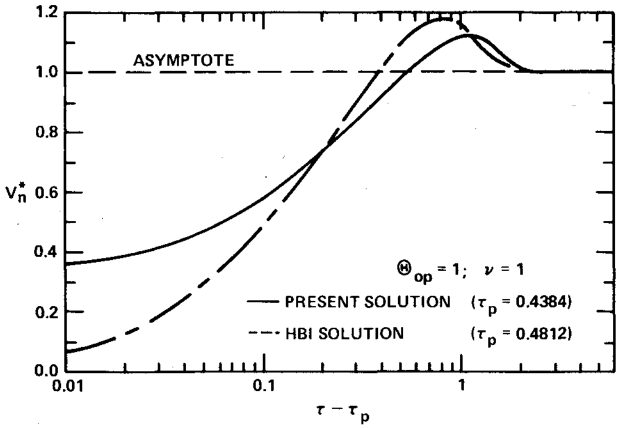


b) **Ablation speed**

Fig. 11 Ablation for $q_0 = Ae^{\tau}$, $\Theta_{0p} = 1$, $\nu = 1/2$.



a) **Ablation thickness**



b) **Ablation speed**

Fig. 12 Ablation for $q_0 = Ae^{\tau}$, $\Theta_{0p} = 1$, $\nu = 1$.

The single-differential equation to be solved numerically is:

$$\frac{\mathrm{d}\Delta}{\mathrm{d}\tau} = \left(1 + \frac{1}{\nu}\right)\frac{1}{\Delta} - \frac{e^{\tau}}{\Theta_{0p}}\frac{1}{\nu} \tag{51}$$

with the initial condition of $\Delta(\tau_p) = (1 - e^{-\tau_p})^{1/2}$.

The ablation thickness is then obtained from

$$\lambda = \frac{1}{1+\nu}\frac{1}{\Theta_{0p}}(e^{\tau} - e^{\tau_p}) - (\Delta - \Delta_p) \tag{52}$$

As $\tau \to \infty$, Eq. (51) requires that

$$\Delta \to \Delta_{\infty} = \Theta_{0p}(1+\nu)e^{-\tau} \tag{53}$$

Equation (52) then gives

$$\lambda \to \lambda_{\infty} = e^{\tau}/(1+\nu)\Theta_{0p} \quad \text{as } \tau \to \infty \tag{54}$$

which checks with the result of the present method, Eq. (43b). It is easy to show that the asymptotic solutions of the ablation speed given by the two methods also agree. It should be noted, however, that the initial ablation speed is predicted differently by the two methods. While the HBI solutions are likely more accurate very near $\tau = \tau_p$, the present solutions are perhaps more accurate for most values of τ away from τ_p by inference from the results discussed in Sec. III.C.

The HBI solutions for $(\Theta_{0p}, \nu) = (½, ½)$, $(½, 1)$, $(1, ½)$, and $(1,1)$ are also included in Figs. 9-12, respectively, for comparison with the corresponding solutions by the present method.

C. Discussion

The comparison between the present solutions and the HBI solutions is very similar to that for the case of a power-law boundary heat flux treated in Sec. III. The characteristic maximum in the normalized ablation speed is again present in both types of solutions. Also, Δ, as predicted by both methods, reaches a maximum in the ablation period before approaching zero in the limit of $\tau \to \infty$, and the behavior is common to both types of solutions.

V. Concluding Remarks

Application of new integral method developed by the author[2-4] has been made to transient ablation problems with a time-dependent heat flux $q_0(t)$. Explicit results are obtained only for two classes of the applied heat flux, i.e., $q_0 \sim t^m$ and $q_0 \sim e^t$. In both cases, the heat flux is a monotonic function of time. Although the method appears equally applicable to general cases of $q_0(t)$, it is expected that the choice of an approximate temperature profile in the case of a non-monotonic $q_0(t)$ would perhaps require some careful considerations.

For lack of published exact solutions to the ablation problems treated in this paper, a direct assessment of the accuracy of the present ablation solution is not possible at this time. However, it appears reasonable to infer the accuracy of these solutions from the merits of the method and the improvements over the classical HBI method demonstrated in the special case of $m = 0$ and in the preablation solutions. It is reiterated here that the quantities of primary interest in these integral solutions are the boundary properties, e.g., λ, V^*, etc., rather than the details of the temperature field.

Acknowledgment

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